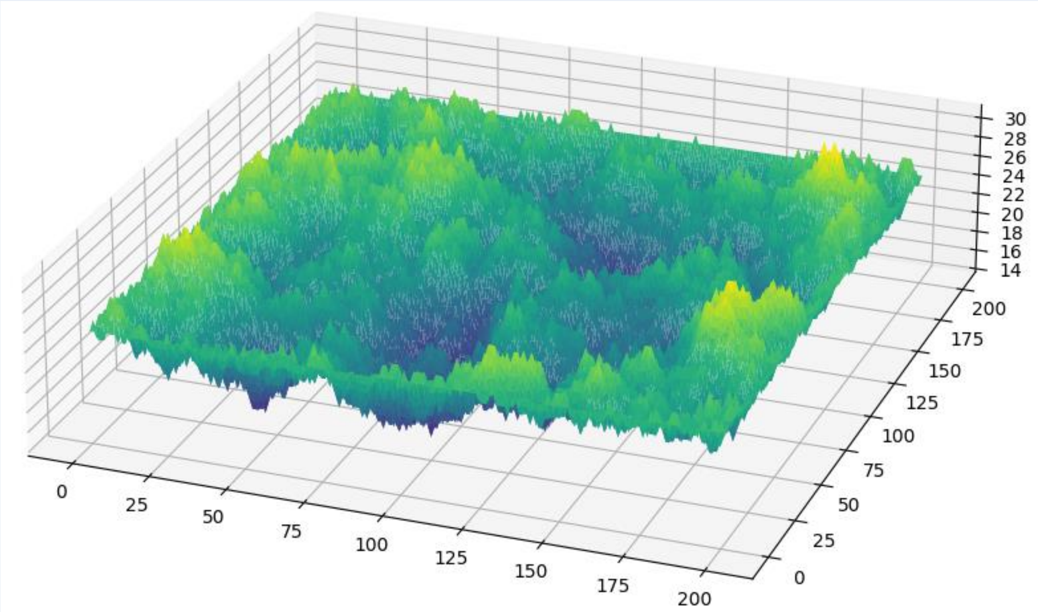
Non-constant ground configurations in the disordered Ising ferromagnet



Ron Peled, Tel Aviv University,
on sabbatical at the IAS and Princeton University

Joint work with Michal Bassan and Shoni Gilboa

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The disordered Ising ferromagnet

- The Ising model is a simple model for a magnet.
  We consider it on the $\mathbb{Z}^D$ lattice, with configurations given by $\sigma: \mathbb{Z}^D \to \{-1,1\}$.
- Given $\eta: E(\mathbb{Z}^D) \to (0,\infty)$, the Hamiltonian of the model is

  $$H^\eta(\sigma) = -\sum_{x\sim y} \eta_{\{x,y\}}\sigma_x\sigma_y$$

  Thus, configurations with more alignment of adjacent spins are energetically preferred. The coupling constants η allow for inhomogeneity in the lattice, assigning different energetic weights to different edges.
- We use the term disordered Ising ferromagnet (or random-bond Ising model) for the case that the (η_e) are (quenched) random, sampled independently from a distribution ν on the non-negative reals (e.g., ν is uniform on $[a,b]$ for $b > a > 0$).
- We wish to understand the low-temperature properties of the disordered Ising model and as a first step we consider its zero temperature properties.
  In other words, we study configurations which minimize H^η in a suitable sense.

Ground configurations

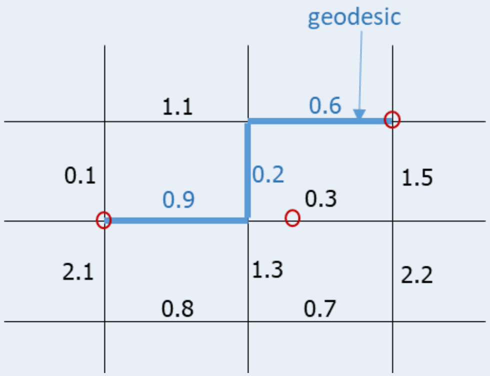
- Reminder: Configurations are $\sigma: \mathbb{Z}^D \to \{-1,1\}$. Given $\eta: E(\mathbb{Z}^D) \to (0,\infty)$, the Hamiltonian is
$$H^\eta(\sigma) = -\sum_{x\sim y} \eta_{\{x,y\}}\sigma_x\sigma_y$$
- Ground configurations: A configuration σ is called a ground configuration if it satisfies that $H^\eta(\sigma) \le H^\eta(\sigma')$ for all configurations σ' which differ from σ at finitely many vertices. Note that while H^η itself is a non-convergent sum, the difference $H^\eta(\sigma) - H^\eta(\sigma')$ is then well defined.
Ground configurations are a kind of local minimizers of H^η.
- It is clear that the two constant configurations are ground configurations.
- Challenge: Does the disordered Ising ferromagnet admit non-constant ground configurations? (this has probability 0 or 1 by ergodicity)
- Discussed in Newman's (1997) book, by Wehr (1997) and by Wehr-Wasielak (2016) (the latter shows that such ground configurations do not arise in a translation-covariant metastate).

Bigeodesics in first-passage percolation

- First-passage percolation models a random perturbation of Euclidean geometry, formed by a random media with short-range correlations (Hammersley-Welsh 1965). We describe the discrete setting of the lattice $\mathbb{Z}^D$.
- Edge weights: Independent and identically distributed non-negative $(\eta_e)_{e \in E(\mathbb{Z}^D)}$.
- Passage time: A random metric $T_{u,v}$ on $\mathbb{Z}^D$ given by

$$T_{u,v} := \min \sum_{e \in p} \tau_e$$

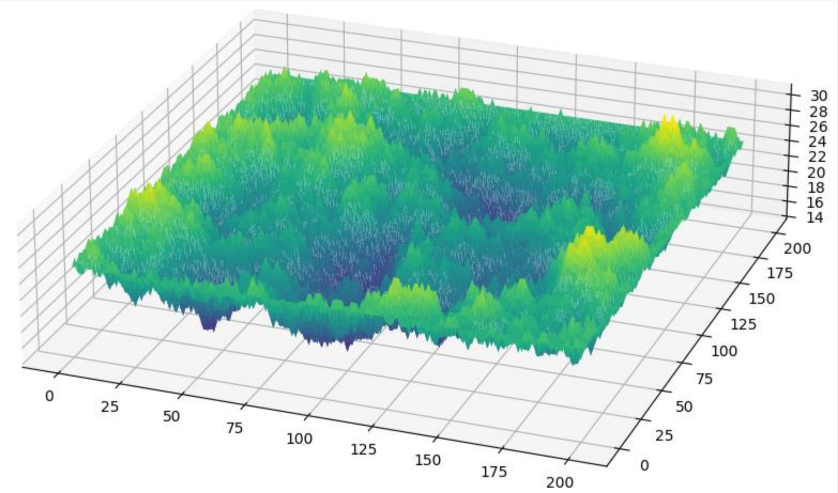
with the minimum over paths p connecting u and v.



- Geodesic: The unique path p realizing $T_{u,v}$, denoted $\gamma_{u,v}$.
- Goal: Understand the large-scale properties of the metric T. In particular, understand the geometry and length of long geodesics.
- Equivalence: One checks (see Newman (1997)) that in dimension $D = 2$, non-constant ground configurations exist in the disordered Ising ferromagnet if and only if infinite bigeodesics exist in first-passage percolation with the same η.
- Conjecture: It is believed that bigeodesics do not exist in dimension $D = 2$. This has been verified under strong unproven assumptions and in related integrable models.

Dobrushin boundary conditions

- Strategy: A natural way to construct non-constant ground configurations is as a limit of finite-volume ground configurations with Dobrushin boundary conditions.
- Consider the disordered Ising ferromagnet in $\Delta_L := \{-L, \dots, L\}^D$. Put boundary values $\sigma_x = \mathrm{sgn}(x_D - 1/2)$ for $x \notin \Delta_L$, where x_D is the last coordinate of x.
- Let $\sigma^{Dob,L}$ be the configuration minimizing H^η with these boundary values.
- Interface: The configuration $\sigma^{Dob,L}$ defines a surface (domain wall/cut) separating the $+1$ spins from the -1 spins. The surface may have overhangs.
- Localization: If the surface "height" above the origin remains tight as $L \to \infty$, then any weak limit of $\sigma^{Dob,L}$ is a (measure on) non-constant ground configurations.
- The fact that the surface delocalizes in dimension $D = 2$ is also called the Benjamini-Kalai-Schramm 2003 midpoint problem.
  This was established by Damron-Hanson 2015 (under an assumption), Ahlberg-Hoffman 2016 (unconditionally) and Dembin-Elboim-P. 2022 (quantitatively).



Main Results

- Theorem (Bassan-Gilboa-P.): Suppose the disorder distribution ν is $\mathrm{Uniform}[a,b]$. Then there exists $D_0(a,b)$ such that for every dimension $D \geq D_0(a,b)$, almost surely, the finite-volume ground configuration $\sigma^{Dob,L}$ converges as $L \to \infty$ to a non-constant ground configuration σ^{Dob}.
  Moreover, $D_0(a,b) = 4$ when the ratio $\frac{b-a}{a}$ is sufficiently small.
- Additionally, the limit configuration σ^{Dob} may be chosen as a measurable $\mathbb{Z}^{D-1}$-translation-covariant function of the disorder η.
- Remarks: 1) The technique applies to a wider class of distributions (Lipschitz functions of Gaussians with compact support in $(0,\infty)$).
  2) A version of the theorem is also established for anisotropic disorder, in which the disorder distribution of the edges in the D'th direction differs from that of the other edges.

The “no overhangs” approximation (a disordered Solid-On-Solid model)

- Bovier-Külske 94,96 studied the problem of interface localization in the “no overhangs”, or height function, approximation. Here, $d := D - 1$.
- Model (Solid-on-Solid height function in a random environment): Configurations are $\varphi: \mathbb{Z}^d \to \mathbb{Z}$. Hamiltonian is

$$H^{SOS,V}(\varphi) := -a \sum_{u \sim v} |\varphi_u - \varphi_v| - \sum_{v} V_v(\varphi_v)$$

where the potentials $\left(V_v(k)\right)_{v,k}$ are independent, distributed as $\text{Uniform}[a, b]$ (or more general distributions).
- Approximation: Obtained from the disordered Ising ferromagnet under Dobrushin boundary conditions under the assumptions that the interface has no overhangs and the coupling constants of all edges except in the D’th direction are exactly a.
- Theorem (Bovier-Külske 1994): For $d \geq 3$, if $\frac{b-a}{a}$ is sufficiently small then, almost surely, the finite-volume ground configurations (or low-temperature measures) with zero boundary values converge to an infinite-volume (localized) measure.
- Theorem (Bovier-Külske 1996): For d=1,2, there are no translation-covariant and “coupling-covariant” low-temperature Gibbs states.

Proof approach

- Bovier-Külske 1994 (86 pages!) adapt the rigorous renormalization approach of Bricmont-Kupiainen 1988 who proved long-range order for the random-field Ising model (RFIM) in dimensions $d \geq 3$ (at weak disorder and low temperature, following Imbrie 1984 at zero temperature).
- Recently, Ding-Zhuang 2021 found a short proof for long-range order in the RFIM in dimensions $d \geq 3$.
  The proof finds a clever way to adapt the Peierls argument to the RFIM setting, using a concentration argument for the ground energy (following a concentration argument of Fisher-Fröhlich-Spencer 1984 for the "no contour within contour" approximation of the RFIM. Also related is Chalker 1983).
- We observe that the argument of Ding-Zhuang 2021 adapts to yield a short proof for the Solid-On-Solid setup.
  Our main result is then obtained by a complicated synthesis of Dobrushin's (1972) proof of the existence of localized interfaces in the $d \geq 3$ pure Ising model with the approach of Ding-Zhuang 2021.

Solid-On-Solid proof I (energy improvement)

- Model and notation: Configurations are $\varphi\colon \mathbb{Z}^d \to \mathbb{Z}$. Hamiltonian is
$$H^{SOS,V}(\varphi) \coloneqq -a\sum_{u\sim v}|\varphi_u - \varphi_v| - \sum_v V_v(\varphi_v)$$
where the potentials $\left(V_v(k)\right)_{v,k}$ are independent, distributed as $\mathrm{Uniform}[a,b]$.
- Let $\varphi^{V,L}$ be the minimizer of $H^{SOS,V}(\varphi)$ in $\left\{\varphi\colon \varphi|_{\Lambda_L^c} \equiv 0\right\}$ with $\Lambda_L \coloneqq \{-L,\dots,L\}^d$. Write $GE^{V,L} \coloneqq H^{SOS,V}\left(\varphi^{V,L}\right)$ so that the differences $GE^{V',L} - GE^{V,L}$ are defined.
- Graph notation: Write $\mathcal{C} \coloneqq \left\{A \subset \mathbb{Z}^d \text{ finite} : A \text{ and } A^c \text{ connected}, 0 \in A\right\}$. Let ∂A be the edge boundary of $A \subset \mathbb{Z}^d$.
- Claim: $\exists \alpha_d > 0$ such that the following holds. If $\varphi_0^{V,L} = k > 0$ then there exists $A \in \mathcal{C}$ and $0 < r \le k$ such that $r|\partial A| \ge k^{\alpha_d}$ and $\varphi_u^{V,L} \ge \varphi_v^{V,L} + r$ when $\{u,v\} \in \partial A$, $u \in A$.
- In this setting, $H^{SOS,V}\left(\varphi^{V,L}\right) - H^{SOS,V^A}\left(\varphi^{V,L} - r1_A\right) \ge ar|\partial A| \ge ak^{\alpha_d}$, where we set $V_v^{A,r}(m) \coloneqq \begin{cases} V_v(m+r) & v \in A \\ V_v(m) & v \notin A \end{cases}$.
- Ground energy improvement: In particular, $GE^{V,L} - GE^{V^{A,r},L} \ge ar|\partial A| \ge ak^{\alpha_d}$.

Solid-On-Solid proof II (energy concentration)

- Ground energy improvement: Recall $V_v^{A,r}(m) := \begin{cases} V_v(m+r) & v \in A \\ V_v(m) & v \notin A \end{cases}$.

  If $\varphi_0^{V,L} = k > 0$ then there exist $A \in \mathcal{C}$ and $0 < r \le k$ such that
  $$GE^{V,L} - GE^{V^{A,r},L} \ge ar|\partial A| \ge ak^{\alpha_d}$$
- Theorem (adapting Ding-Zhuang 2021, following Fisher-Fröhlich-Spencer 1984):

  Let $d \ge 3$. Let $\frac{b-a}{a}$ be sufficiently small. For each $M, r, t > 0$, if $t^2 \ge C_d M^{\frac{d}{d-1}}$ then
  $$\mathbb{P}\left(\exists A \in \mathcal{C}, |\partial A| = M, \left|GE^{V,L} - GE^{V^{A,r},L}\right| \ge t\right) \le C_d e^{-c_d t^2 M^{-\frac{d}{d-1}}}.$$
- Basic concentration (two-point estimate): For each $A, B \subset \Lambda_L, r \in \mathbb{Z}$,
  $$\mathbb{P}\left(\left|GE^{V^{A,r},L} - GE^{V^{B,r},L}\right| \ge t\right) \le Ce^{-c\frac{t^2}{|A\Delta B|}}.$$
- This is a consequence of the inequality
  $$\mathbb{P}\left(\left|GE^{V^{A,r},L} - \mathbb{E}\left(GE^{V^{A,r},L} \,\middle|\, V|_{(A\Delta B)^c}\right)\right| \ge t \,\middle|\, V|_{(A\Delta B)^c}\right) \le Ce^{-c\frac{t^2}{|A\Delta B|}} \quad (1)$$
  and the fact that $\mathbb{E}\left(GE^{V^{A,r},L} \,\middle|\, V|_{(A\Delta B)^c}\right) = \mathbb{E}\left(GE^{V^{B,r},L} \,\middle|\, V|_{(A\Delta B)^c}\right)$. Inequality (1) follows from Hoeffding's inequality (resampling a column of disorder at once) or from concentration of Lipschitz functions of Gaussian random variables.

Solid-On-Solid proof III (coarse graining)

- Theorem (adapting Ding-Zhuang 2021, following Fisher-Fröhlich-Spencer 1984): Let $d \geq 3$. Let $\frac{b-a}{a}$ be sufficiently small. For each $M, r, t > 0$, if $t^2 \geq C_d M^{\frac{d}{d-1}}$ then

$$\mathbb{P}\left(\exists A \in \mathcal{C}, |\partial A| = M, \left|GE^{V,L} - GE^{V^{A,r},L}\right| \geq t\right) \leq C_d e^{-c_d t^2 M^{-\frac{d}{d-1}}} \quad (2)$$

- Basic concentration (two-point estimate): For each $A, B \subset \Lambda_L, r \in \mathbb{Z}$,

$$\mathbb{P}\left(\left|GE^{V^{A,r},L} - GE^{V^{B,r},L}\right| \geq t\right) \leq Ce^{-c\frac{t^2}{|A\Delta B|}} \quad (3)$$

- The inequality (3) yields the bound on the right-hand side of (2) for a fixed A, using that $|A| \leq C_d M^{\frac{d}{d-1}}$. However, this does not suffice to conclude by a union bound over all A, since there are $\approx e^{C_d M}$ such A and we need the case $t \approx M$.
- The proof uses a chaining argument using (3), following a chaining scheme introduced by Fisher-Fröhlich-Spencer 1984. The idea is to coarse grain the possible sets A, defining the mth approximation A^m of A as follows:
- Partition $\mathbb{Z}^d$ into cubes of side length 2^m. Put a cube C in A^m if $|C \cap A| \geq \frac{1}{2}|C|$.
- In this way we obtain a sequence of sets $A = A^0, A^1, \dots, A^{m_0}$, with m_0 chosen so that a union bound is applicable to the set of all possible A^{m_0} and to the "transitions" from each A^m to A^{m-1}.

Disordered Ising ferromagnet adaptations

- There are many difficulties in adapting the proof from the Solid-On-Solid model to the Dobrushin interface of the disordered Ising ferromagnet.

1. Instead of shifting φ and V on a single set A, we have to consider a more general shift function $s: \mathbb{Z}^d \to \mathbb{Z}$ which tells how much to shift each column of the Ising configuration σ and the disorder η.
   This necessitates a development of the corresponding enumeration and coarse (and fine) graining techniques for such shift functions.
2. To obtain a shift function leading to energy improvement we need to rely on Dobrushin's (1972) decomposition of the interface into walls. However, in regions with overhangs this technology doesn't directly yield a shift. To overcome this, we found and proved the following combinatorial fact:

   Lemma: In the disordered ferromagnet with disorder $\eta: E(\mathbb{Z}^d) \to [0, \infty)$, if η is constant on the edges $\{x, x + e_i\}$ with $i \neq D$, then (one of) the zero-temperature interface under Dobrushin boundary conditions has no overhangs.
   This also allows to see the Solid-On-Solid setup as a special case of the disordered ferromagnet.
3. A serious complication arises from the fact that having overhangs in the interface leads to a weaker concentration bound (larger Lipschitz constant). To overcome this, we employ an involved induction scheme over the energetic improvement.

Open questions

- "Wide spread" disorder: For $\mathrm{Uniform}[a, b]$ disorder distribution, our results prove localization of the Dobrushin interface when $D \geq 4$ and $\frac{b-a}{a}$ is small.
  What happens for other choices of $\frac{b-a}{a}$?
  Based on considerations of minimal surfaces in random environment with continuous values, we conjecture that there is always localization for $D \geq 6$.
  It may be the case that for $D = 4,5$ (or just $D = 4$) there is a roughening transition as $\frac{b-a}{a}$ grows, from a localized to a delocalized regime. Related conjecture for the simpler "Integer-valued random-field Gaussian free field" is in Dario-Harel-P. 2023.
- Dimension D=3: As in the Bovier-Külske 1996 result, we expect that the Dobrushin boundary conditions interface is always delocalized. We even expect its height to be a power of L. However, non-constant ground configurations may still exist.
- Uniqueness: We prove the existence of a $\mathbb{Z}^{D-1}$-translation-covariant non-constant ground configuration in the disordered Ising ferromagnet. We believe that such a ground configuration is unique up to translations in the D'th coordinate direction.
- Tilted surfaces: What happens under "tilted" Dobrushin boundary conditions? Still expect localization for given tilt in $D \geq 6$.